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# On skew $C_1C_2$-Symmetric operators

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ABSTRACT

Let $C_1$ and $C_2$ be conjugation operators, both of which are antilinear, isometric, and involution mappings, defined on a separable complex Hilbert space $\mathcal{H}$. This paper introduces the concept of skew $C_1C_2$-symmetric operators ($skew\ C_1C_2$-S.O). A bounded linear operator A on $\mathcal{H}$ is classified as a $skew\ C_1C_2$-S.O. if it satisfies the condition ( $C_1A = -A^* C_2$ ), or equivalently, ( $A = -C_1 A^* C_2$ ). We examine and analyze several fundamental properties of such operators and provide a concrete example to illustrate this notion.

## 1. INTRODUCTION

An algebra to all bounded linear operator specified on a separable complex Hilbert space $\mathcal{H}$ is represented by the notation B($\mathcal{H}$). A conjugation operation on $\mathcal{H}$ is antilinear operator $C:\mathcal{H} \to \mathcal{H}$ that fulfills for any $x, y \in \mathcal{H}$ and property of involution ($C^2 = 1$), and $\langle Cx, Cy\rangle = \langle x, y\rangle$. The research of complex symmetric operation was started in 2005 by [1]. According to their definition, an operator $A \in$ B($\mathcal{H}$) is considered $C$-symmetric if $CA = A^*C(A = CA^*C)$; it is complex symmetric; it is $C$-symmetric with regard to some $C$ [1,2].

The idea of symmetric matrices in linear algebra are generalized by complex symmetric operation. Since for $x, y \in \mathcal{H}$, the matrix of $C$-symmetric operator $A$ with regard to $\{e_n\}$ is symmetric. This because if $C$ is a conjugation on $\mathcal{H}$, then there is an orthonormal basis $\{e_n\}$ of $\mathcal{H}$ in order to $Ce_n = e_n$ to all $n$ [1]. The opposite is also true. In other words, $A$ is complex symmetric if there is an orthonormal basis such that $A$ has a symmetric matrix representation [1]. If there is a conjugation $C$ on $\mathcal{H}$ in order to $CA = -A^*C(A = -CA^*C)$, then an operator $A \in$ B($\mathcal{H}$)is skew complex symmetric.

M. putinar, and W.Wogen, in different combinations, conducted a general investigation of such operators in [1-12]. The idea of a complex symmetric operation was expanded by Dakheel and Ahmed [13] in 2022. They defined a $C_1C_2$-S.O. as one in which an operator $A \in$ B($\mathcal{H}$) has certain conjugations $C_1$ and $C_2$ on $\mathcal{H}$ such that $C_1A = A^*C_2(A = C_1A^*C_2)$.

The definition of skew complex symmetric operation is expanded in this work to be as follows: if $C_1A = -A^*C_2(A = -C_1A^*C_2)$, then a bounded linear operator. We look at basic characteristics of these operators and arrive at the following conclusion: there are two orthonormal basis of $\mathcal{H}$ relative to that's $A$ acknowledges a symmetric matrix illustration if $A$ is a skew $C_1C_2$-S.O. . Additionally, we examine the matrix of skew $C_1C_2$-S.O. and demonstrate a few of its uses. Additionally, we looked at the skew $C_1C_2$-S.O. tensor product, direct sum, and tensor sum.

## 2. Main Results

This part deals with to introduce the concept of skew $C_1C_2$-S.O. , which serve as a generalization of skew complex symmetric operation. Furthermore, we examine fundamental properties of this concept and investigate its theoretical implications.

**Definition 2.1 :** put $\mathcal{H}$ be separable, complex Hilbert space and let $C_1$ and $C_2$ be conjugate linear operators acting on $\mathcal{H}$ ($C_1 \neq C_2$) that are both involution ($C_1^2 = C_2^2 = I$) and isometric ( $\langle C_1x, C_1y\rangle = \langle y, x\rangle$ and $\langle C_2s, C_2k\rangle = \langle k, s\rangle$ to $all\ x, y, s$ and $k$ in $\mathcal{H}$). A B.L.O. $A$ on $\mathcal{H}$ is

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claimed to be $C_1C_2$-S.O. if met the condition $C_1$A = -A$^*$ C$_2$ equivalently expressed as (A = -$C_1$ A$^*$C$_2$).

**Example 2.2:** Let $\mathcal{C}_1 = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}$ and $\mathcal{C}_2 = \begin{bmatrix} C_1 & 0 \\ 0 & -C_2 \end{bmatrix}$ are conjugations on $\mathcal{H} \oplus \mathcal{H}$ such that $C_1$, $C_2$ are conjugations operator on $\mathcal{H}$ and let $\mathcal{S}$ be skew complex symmetric operator such that $C_1\mathcal{S} = -\mathcal{S}^*C_1$. Then A = $\begin{bmatrix} \mathcal{S} & 0 \\ 0 & 0 \end{bmatrix}$ is skew $C_1C_2$-S.O. .

**Remarks 2.3:** For Conjugations $C_1$ and $C_2$, the following statements are holds:

1. Every skew complex symmetric operator is skew $C_1C_2$-S.O..
2. Put $A$ is skew $C_1C_2$-S.O. , then A= -C$_2$A$^*$C$_1$.
3. Put $A$ is skew $C_1C_2$-S.O. , then so is A$^*$.
4. Put $A$ is skew $C_1C_2$-S.O. , then so is A$^{-1}$.
5. The set of skew $C_1C_2$-S.O. is a subspace of B($\mathcal{H}$) for the same conjugations $C_1$ and $C_2$.

**Proof:**

1. Let A be a skew $C_1C_2$-S.O. such that $C_1$ A = -A$^*$C$_2$. Assume that $C_1$ = C$_2$, hence A is skew complex symmetric operator.
2. Assume that A is skew $C_1C_2$-S.O. satisfying the condition A = - $C_1$ A$^*$C$_2$ equivalently expressed as (-$C_1$ A C$_2$= A$^*$), to demonstrate that A= - C$_2$ A$^*$$C_1$ we have: <-C$_2$ A$^*$ $C_1 w, z$ > = <C$_2 z$, -C$_2$ C$_2$ A$^*$$C_1$ $w$ > = < -A C$_2$ $z$, $C_1$ $w$ >= < $C_1C_1w$, -C$_1$ A C$_2$ $z$ > = < $w$, A$^*$ $z$ > = <A $w$, $z$ > , for all $w, z \in \mathcal{H}$.

3. Suppose that A skew $C_1C_2$-S.O. , such that $C_1$ A = -A$^*$C$_2$ (A$^*$ = -$C_1$ AC$_2$).

To show that C$_1$A$^*$ = -AC$_2$ :
< $C_1$ A$^*$ $w$, $z$ > = < $C_1$ $z$, $C_1$ $C_1$ A$^*$ $w$ > = < $C_1 z$, A$^*$ $w$ >= < $C_1$ $z$, -$C_1$ A C$_2$ $w$ > = < -$C_1$ ($C_1$ A C$_2$ $w$), $C_1$ $C_1$ $z$ > = < -AC$_2$ $w$, $z$ > , for all $w, z \in \mathcal{H}$.

4. Let A = - C$_1$ A$^*$ C$_2$ , to show that A$^{-1}$ = -C$_1$ (A$^{-1}$)$^*$C$_2$: < -$C_1$ (A$^{-1}$)$^*$C$_2$ $w$, $p$ > = < $C_1$ $p$, -$C_1$ $C_1$ (A$^{-1}$)$^*$ C$_2$ $w$ > = < $C_1 p$, - (A$^{-1}$)$^*$ C$_2$ $w$ >= < -A$^{-1}$ $C_1$ $p$, C$_2$ $w$ > = < C$_2$ C$_2$ $w$, -C$_2$ A$^{-1}$ $C_1$ $p$ >= < $w$, -($C_1$A C$_2$)$^{-1}$ $p$ >= < $w$, (A$^*$)$^{-1}$ $p$ > = < $w$,- (-A$^{-1}$)$^*$ $p$ > = < A$^{-1}$ $w$, $p$ > , for all $w, p \in \mathcal{H}$.

5. The proof requires two steps:

i. if A$_1$, A$_2$ are skew $C_1C_2$-S.O. , then so is A$_1$+ A$_2$.

ii. if A skew C$_1$C$_2$-symmetric and $b \in \mathbb{C}$, then $b$ A is skew C$_1$C$_2$-symmetric.

For the first portion, observe that:
Since A$_1$, A$_2$ are skew C$_1$C$_2$-symmetric, it follows that $C_1$ A$_1$ = -A$_1^*$C$_2$ and $C_1$ A$_2$ = - A$_2^*$C$_2$.
To prove that A$_1$+A$_2$ is skew $C_1C_2$-S.O. , $C_1$ (A$_1$+A$_2$) = $C_1$ A$_1$ + $C_1$ A$_2$ = -A$_1^*$C$_2$ -A$_2^*$C$_2$ = - (A$_1$+A$_2$)$^*$ C$_2$. Thus (1) holds. For ii, since A skew C$_1$C$_2$-symmetric, it follows that $C_1$ A = -A$^*$C$_2$ and $b \in \mathbb{C}$.

To show that $b$ A skew $C_1$ C$_2$-symmetric for the same conjugations $C_1$, C$_2$, $C_1$ ($b$A) = $\overline{b}$ $C_1$ A = - $\overline{b}$ A$^*$C$_2$ =-($b$A)$^*$ C$_2$.

The following proposition gives useful characterizations of skew $C_1C_2$-S.O. .

**Proposition 2.4:** take $A$ in B($\mathcal{H}$) then an operator $A$ in B($\mathcal{H}$) is skew $C_1C_2$-S.O. iff there are two orthonormal bases of $\mathcal{H}$ with that's $A$ has symmetric matrix illustration.

**Proof:** Let A be skew $C_1C_2$-S.O. such that $C_1$ A= -A$^*$C$_2$ and let $\{u_n\}$ and $\{v_n\}$ are two orthonormal bases such that C$_1 u_n = u_n$ and C$_2$ $v_m = v_m$ for all $n, m \in \mathbb{N}$. The matrix of skew $C_1C_2$-S.O. A with respect to $\{u_n\}$ and $\{v_n\}$ is skew symmetric, to show that:

[A]$_{ij}$ = <A$v_j$, $u_i$>
= <- $C_1$ A$^*$C$_2$$v_j$, $u_i$>
= <- $C_1$ A$^*$$v_j$, $u_i$>
= - <$C_1 u_i$, $C_1$ $C_1$A$^*$$v_j$>
= - < $C_1 u_i$, A$^*$$v_j$>
= - <$u_i$, A$^*$$v_j$>
= - <A$u_i$, $v_j$>
= $-[A]_{ji}$ , for $1 \le i \le n, 1 \le j \le m$.

Conversely, let $\{u_n\}$ and $\{v_m\}$ be two orthonormal bases such that C$_1 u_n = u_n$ and C$_2$ $\sigma_m = \sigma_m$ for all $n, m \in \mathbb{N}$. Define the conjugations C$_1$ and C$_2$ by C1($\sum_n a_n u_n$)= $\sum_n \overline{a_n} u_n$, C2($\sum_m c_m \sigma_m$)= $\sum_m \overline{c_m} \sigma_m$.

By hypothesis, the matrix ensures that $<A u_n, \sigma_m> = - <A\sigma_m, u_n>$ for all $n, m \in \mathbb{N}$, to show that A is skew $C_1C_2$-S.O. in order to $- C_1A^* C_2 = A$.

$$< - C_1 A^*C_2\, \sigma_m, u_n> = < - C_1 A^*\, \sigma_m, u_n>$$
$$= < C_1\, u_n, - C_1 C_1 A^*\, \sigma_m>$$
$$= < C_1\, u_n, - A^*\, \sigma_m>$$
$$= < u_n, - A^*\, \sigma_m>$$
$$= < -A u_n, \sigma_m >$$
$$= - < A u_n, \sigma_m >$$
$$= < A\, \sigma_m, u_n>.$$

**Proposition 2.5:** If $\{A_n\}$ be a sequence of skew $C_1C_2$-S.O. with the same conjugations $C_1$ and $C_2$ in order to $\lim_{n\to\infty} ||A_n - A|| = 0$, then A is also skew $C_1C_2$-S.O..

**Proof:** Let $\{A_n\}$ be a sequence of skew $C_1C_2$-S.O. operator such that $C_1A_n = -A_n^* C_2$ ($A_n = - C_1A_n^*C_2$) for the same conjugations $C_1$ and $C_2$ with $\lim_{n\to\infty} ||A_n - A|| = 0$, we must prove that $A= - C_1A^* C_2$:

$$\| -A- C_1A^* C_2\| \le \| -A - C_1A_n^* C_2\| + \| C_1A_n^* C_2 - C_1A^* C_2 \|$$
$$\le \| -A + A_n \| + \|C_1\| \, \| A_n^* - A^*\| \, \|C_2\|$$

Since $\|C_1\| = \|C_2\|=1$, then

$$\le \| A_n - A\| + \| A_n^* - A^* \|$$
$$\le \| A_n - A\| +\| A_n - A\|$$
$$\le 2 \| A_n - A\|$$

Which tends to zero as $n\to \infty$. Hence skew $C_1C_2$-S.O..

The proper notion of equivalence for skew $C_1C_2$-S.O. is unitary equivalence as the following shows:

**Proposition 2.6:** If $A_1\in B(\mathcal{H}_1)$ is skew $C_1C_2$-S.O. and $U{:}\,\mathcal{H}_1 \to \mathcal{H}_2$ is unitary operator, then there exists $A_2\in B(\mathcal{H}_2)$ is skew $C_3C_4$-S.O. such that $A_2 = UA_1 U^*$, $C_3 = UC_1U^*$, $C_4 = UC_2U^*$.

**Proof:**

Since $A_1$ is skew $C_1C_2$-S.O. such that $C_1A= -A^*C_2$, then we have:

$$C_3A_2 = (UC_1U^*)(UA_1U^*)$$
$$= (U\text{-}A_1^*U^*)(UC_2U^*)$$
$$= - (UA_1U^*)^*C_4$$
$$= - A_2^* C_4.$$

**Proposition 2.7:** assume that the Cartesian decomposion of $A = \mathcal{X} + i\mathcal{Y}$ if and only if both $X$ and $Y$ are skew $C_1C_2$-S.O. with regard to identical conjugations $C_1$ and $C_2$, then $A$ is skew $C_1C_2$-S.O.

**Proof:** Let A be skew $C_1C_2$-S.O. such that $C_1A = -A^*C_2$ with $A = \mathcal{X} + i\mathcal{Y}$ and $\mathcal{X} = \frac{1}{2}(A + A^*)$ and $\mathcal{Y}= \frac{1}{2i}(A - A^*)$.

To show that $\mathcal{X}$ and $\mathcal{Y}$ are skew $C_1C_2$-S.O. with the same conjugation $C_1$, $C_2$, then we have:

$$C_1\, \mathcal{X} = C_1 (\tfrac{1}{2} (A + A^*))$$
$$= \tfrac{1}{2} (C_1A + C_1A^*)$$
$$= \tfrac{1}{2} (-A^*C_2 - AC_2)$$
$$= - \tfrac{1}{2} (A^* + A)\, C_2$$
$$= - \mathcal{X}^*\, C_2.$$

Similarly, we deduse that $\mathcal{Y}$ is also skew $C_1C_2$-S.O.. Conversely, since $\mathcal{X}$ and $\mathcal{Y}$ are skew $C_1C_2$-S.O. operators with respect to the same conjugations $C_1$ and $C_2$, then we obtain directly $C_1A = -A^*C_2$.

## 3.Tensor product and direct sum of skew $C_1C_2$-S.O.

This part deals with the necessary condition for the one rank operator on $\mathcal{H}$ to be skew complex $C_1C_2$-symmetric. Moreover, we discuss the tensor product and direct sum of skew complex $C_1C_2$-*S.O.*.

This part starts by the subsequent lemma [14]:

**Lemma 3.1:** Let $C_1$ and $C_2$ be a conjugations on $\mathcal{H}$ and x, y $\in \mathcal{H}$ . Then $C_1(x\otimes y)C_2 = C_1x \otimes C_2y$ on $\mathcal{H}$.

The next proposition shows that when the finite rank operator becomes skew $C_1C_2$-S.O..

**Proposition 3.2:** If A is constant multiple of $-C_1x\otimes -C_2y$, then A is skew $C_1C_2$-S.O..

**Proof**

By previous lemma, we have $C_1(x\otimes y)C_2 = C_1x \otimes C_2y$ for conjugations operators $C_1$ and $C_2$ on $\mathcal{H}$. Then we have:

$C_1AC_2 = C_1(-C_1x\otimes -C_2y)C_2 = - (y\otimes x) = -A^*$. Hence, A is skew $C_1C_2$-S.O..

**Proposition 3.3:** If $A_1$ is complex skew $C_1C_2$-S.O. on $\mathcal{H}_1$ and $A_2$ is skew $C_3C_4$-S.O. on $\mathcal{H}_2$ for some conjugations $C_1$, $C_2$, $C_3$ and $C_4$, then $A_1\otimes A_2$ is skew $(C_1\otimes C_3)(C_2 \otimes C_4)$-symmetric on $\mathcal{H}_1 \otimes \mathcal{H}_2$.

**Proof:** Since $A_1$ is skew $C_1C_2$-S.O. on $\mathcal{H}_1$ and $A_2$ is skew $C_3C_4$-S.O. on $\mathcal{H}_2$, then $C_1A_1 = - A_1^*C_2$ and $C_3A_2 = -A_2^*C_4$.

Now, to show that $A_1\otimes A_2$ is skew $(C_1\otimes C_3)(C_2 \otimes C_4)$-symmetric operator:

$$(C_1\otimes C_3)\,(A_1\otimes A_2) = C_1A_1 \otimes C_3A_2$$
$$= - A_1^*C_2 \otimes -A_2^*C_4$$
$$= - (A_1^*\otimes A_2^*)\,(C_2\otimes C_4)$$
$$= - (A_1 \otimes A_2)^*\,(C_2 \otimes C_4).$$

Hence, we get what we want.

**Proposition 3.4:** If $\mathcal{M}$ is skew $C_1C_2$-S.O. on $\mathcal{H}_1$ and $\mathcal{B}$ is skew $C_3C_4$-S.O. on $\mathcal{H}_2$ for some conjugations $C_1$, $C_2$, $C_3$ and $C_4$ , then $\mathcal{M} \oplus \mathcal{B}$ is skew $(C_1\oplus C_3)(C_2\oplus C_4)$-symmetric on $\mathcal{H}_1 \oplus \mathcal{H}_2$.

**Proof:**

Let $\mathcal{M}$ be a skew $C_1C_2$-S.O. on $\mathcal{H}_1$ and $\mathcal{B}$ be a skew $C_3C_4$-S.O. on $\mathcal{H}_2$.

$$(C_1\oplus C_3)(\mathcal{M}\oplus\mathcal{B}) = C_1\mathcal{M}\oplus C_3\mathcal{B}$$
$$= -\mathcal{M}^*C_2\oplus -\mathcal{B}^*C_4$$
$$= -(\mathcal{M}^*\oplus\mathcal{B}^*)(C_2\oplus C_4)$$
$$= -(\mathcal{M}\oplus\mathcal{M})^*(C_2\oplus C_4).$$

**Proposition 3.5:** If $\mathcal{M}$, $\mathcal{P}$are a skew $C_1C_2$-S.O. on $\mathcal{H}$ and $\mathcal{E}_1$, $\mathcal{E}_2$ are skew $C_3C_4$-S.O. on $\mathcal{H}$, then $(\mathcal{M}\oplus\mathcal{P})\otimes(\mathcal{E}_1\oplus\mathcal{E}_2)$ is skew $(C_1\otimes C_3)(C_2\otimes C_4)$-symmetric operator on $\mathcal{H}\otimes\mathcal{H}$.

**Proof:**

$(C_1\otimes C_3)[(\mathcal{M}\oplus\mathcal{P})\otimes(\mathcal{E}_1\oplus\mathcal{E}_2)] = (C_1\otimes C_3)[\mathcal{M}\otimes\mathcal{E}_1+\mathcal{P}\otimes\mathcal{E}_1+\mathcal{M}\otimes\mathcal{E}_2+\mathcal{P}\otimes\mathcal{E}_2]$

$= (C_1\otimes C_3)(\mathcal{M}\otimes\mathcal{E}_1) + (C_1\otimes C_3)(\mathcal{P}\otimes\mathcal{E}_1) + (C_1\otimes C_3)(\mathcal{M}\otimes\mathcal{E}_2) + (C_1\otimes C_3)\mathcal{P}\otimes\mathcal{E}_2)$

$= (C_1\mathcal{M}\otimes C_3\mathcal{E}_1) + (C_1\mathcal{P}\otimes C_3\mathcal{E}_1) + (C_1\mathcal{M}\otimes C_3\mathcal{E}_2) + (C_1\mathcal{P}\otimes C_3\mathcal{E}_2)$

$= (-\mathcal{M}^*C_2\otimes\mathcal{E}_1^*C_4) + (-\mathcal{P}^*C_2\otimes -\mathcal{E}_1^*C_4) + (-\mathcal{M}^*C_2\otimes -\mathcal{E}_2^*C_4) + (-\mathcal{P}^*C_2\otimes -\mathcal{E}_2^*C_4)$

$= (-(\mathcal{M}^*\otimes\mathcal{E}_1^*)(C_2\otimes C_4)) + (-(\mathcal{P}^*\otimes\mathcal{E}_1^*)(C_2\otimes C_4)) + (-(\mathcal{M}^*\otimes\mathcal{E}_2^*)(C_2\otimes C_4)) + (-(\mathcal{P}^*\otimes\mathcal{E}_2^*)(C_2\otimes C_4))$

$= -((\mathcal{M}^*\otimes\mathcal{E}_1^*) + (\mathcal{P}^*\otimes\mathcal{E}_1^*) + (\mathcal{M}^*\otimes\mathcal{E}_2^*) + (\mathcal{P}^*\otimes\mathcal{E}_2^*))(C_2\otimes C_4)$

$= -((\mathcal{M}^*\oplus\mathcal{P}^*)\otimes(\mathcal{E}_1^*\oplus\mathcal{E}_2^*))(C_2\otimes C_4)$

$= -((\mathcal{M}\oplus\mathcal{P})\otimes(\mathcal{E}_1\oplus\mathcal{E}_2))^*(C_2\otimes C_4).$

**Proposition 3.6:** If $\mathcal{M}_1$ and $\mathcal{M}_2$ are skew $C_1C_2$-S.O. on $\mathcal{H}$ with $C_1\otimes C_2 = C_2\otimes C_1$, then $\mathcal{M}_2\boxplus\mathcal{M}_2$ is a skew $(C_1\otimes C_2)(C_2\otimes C_2)$-symmetric operator on $\mathcal{H}\otimes\mathcal{H}$.

**Proof:**

$(C_1\otimes C_2)(\mathcal{M}_1\boxplus\mathcal{M}_2)^*(C_2\otimes C_2) = (C_1\otimes C_2)(\mathcal{M}_1^*\otimes I + I\otimes\mathcal{M}_2^*)(C_2\otimes C_2)$

$= [(C_1\otimes C_2)(\mathcal{M}_1^*\otimes I) + (C_1\otimes C_2)(I\otimes\mathcal{M}_2^*)](C_2\otimes C_2)$

$= ((C_1\otimes C_2)(\mathcal{M}_1^*\otimes I) + (C_2\otimes C_1)(I\otimes\mathcal{M}_2^*))(C_2\otimes C_2)$

$= (C_1\mathcal{M}_1^*\otimes C_2 + C_2\otimes C_1\mathcal{M}_2^*)(C_2\otimes C_2)$

$= (C_1\mathcal{M}_1^*\otimes C_2)(C_2\otimes C_2) + (C_2\otimes C_1\mathcal{M}_2^*)(C_2\otimes C_2)$

$= C_1\mathcal{M}_1^*C_2\otimes I + I\otimes C_1\mathcal{M}_2^*C_2$

$= -\mathcal{M}_1\otimes I + I\otimes -\mathcal{M}_2$

$= -(\mathcal{M}_2\boxplus\mathcal{M}_2).$

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Arabic Abstract

لتكن $C_1$ و $C_2$ مؤثرات مترافقة معرفة على فضاء هلبرت العقدي القابل للفصل $\mathcal{H}$ . في هذا البحث، قدمنا مفهوم المؤثر المتناظر من النمط skew $C_1C_2$ على أنهُ : المؤثر المُقيد الخطي المُعرف على فضاء هلبرت العقدي القابل للفصل أذا تحقق $(C_1A=-A^*C_2(A=- C_1 A^*C_2)$. أيضاً، تمت دراسة و مناقشة العديد من الخواص و أيضاً أعطاء بعض الأمثلة التي تخص هذا النوع من المؤثرات.