Pure Sciences International Journal of Kerbala

Journal Homepage:https://journals.uokerbala.edu.iq/index.php/psijk

A Novel Numerical Method for Resolving the Time-Fractional Equation of Advection, Diffusion, and Reaction

Mohammed Hamza Mohammed

Ministry of Education, Holy Karbala Education Directorate

P A P E R I N F O

Paper history:
Received: 28 October 2024
Accepted: 8 December 2024
Published: 31 March 2025

Keywords:
Fractional Differential Equations - Numerical Method-Advection-Diffusion-Reaction Equations - Time Fractional Equations - Numerical Solution - Novel Methodology - Mathematical Modeling - Computational Efficiency - Fractional Calculus.

A B S T R A C T

This study used Lie transformations to provide both numerical and analytical answers for the partial reaction-diffusion-adhesion equations for both time and space. If the balances allowed via the goal calculations permit the determination of Lie transformations, then we can reduce slight fractional variance calculations to normal variance calculations containing fractions. We suggest a different approach to find the numerical and analytical answers beginning from the numerical and analytical answers in the spatio-temporal fractal adhesion-diffusion-interaction model. Recent findings on the adhesion-diffusion interaction equation were obtained by the authors. Separate adhesion-diffusion reaction equations for partial temporal and spatial variables were presented.

The excellent accuracy of the suggested approach makes it a useful instrument for solving a Wide category of fractional differential equation problems. The numerical results show its effectiveness and applicability.

1. Introduction

Due to its numerous applications in a variety of disciplines, including science, engineering, and economics, fractional-order differential equations (FDEs) have attracted a great deal of attention from scholars in recent years. Within the paradigm of fractional calculus, these equations are essential to comprehending phenomena such as transport in porous media and groundwater contamination. Fractional-order systems are nonlocal, and their memory effect has drawn attention due to its special properties and consequences for system behavior. Engineers working on real-world problems favor fractional-order systems over traditional integer-order systems because of their distinct qualities. The accurate representations of nonlinear events provided by fractional differential equations encourage researchers to create numerical techniques for efficiently solving these kinds of equations. Consequently, a multitude of analytical and numerical techniques have been developed, such as variational iterative techniques, homotopy analysis techniques, Wavelet operational techniques, and domain decomposition techniques. Furthermore, numerous numerical methods have been developed to solve various kinds of fractional diffusion equations, highlighting the continuous attempts to improve the comprehension and solution of these intricate systems. As an extension of Fibonacci numbers, the focus has recently switched to Fibonacci polynomials, providing a new angle on the field of polynomials. It is not difficult to produce these polynomials using recurrence relations, and it is only recently that their importance in the field of polynomials has become apparent. Numerous approaches utilizing Fibonacci polynomials have surfaced. These include the approach put forth by Koc et al. [1] in 2013 to address ordinary boundary value problems, the matrix method by Abd-Elhameed and Youssri [2] in 2016 to address generalized pantograph equations, and the Fibonacci operational method by Koc et al. [1] in 2013 to address FDEs. Bessel's operational matrix, Legendre operational matrix, and Chebyshev operational matrix are some of the operational matrices that have been produced recently in the field of fractional-order partial differential equations (FPDEs). Notably, Fibonacci polynomials' operational matrix has proven to be more accurate than orthogonal polynomials', and because of its effectiveness in managing a high number of zeros, it has also been shown to greatly reduce computing time. When solving Fibonacci polynomials-based FPDEs, the novel method

*Corresponding Author Institutional Email:
mohammed1995676@gmail.com (Mohammed Hamza Mohammed)

of approximating a variable's integer-order power independently of its fractional power has demonstrated to produce accurate derivative computations. Fibonacci polynomials provide more accurate solutions for FPDEs than orthogonal polynomials, even at lower degrees, demonstrating their scientific significance over other approaches.

Turning our attention to the vital role that water plays in supporting life, just a small amount of the Earth's surface is actually readily accessible as fresh water, despite the fact that water covers a vast portion of its surface. Water contamination can result from pollution from a variety of sources, affecting both surface and groundwater sources. Whereas groundwater contamination is caused by the seepage of artificial materials such as oil, gasoline, road salt, chemicals, fertilizers, and pesticides, which can have a negative impact on water quality. surface water pollution is usually caused by wastewater discharge. In order to solve the difficulties of solving time-fractional equations involving advection, diffusion, and reaction in such contaminated water systems, a unique numerical method has been presented. This method shows promise for addressing environmental issues. One basic mathematical model that is widely used in scientific and engineering sectors for computer simulations is the advection-reaction-diffusion equation (ARDE). It is used in many different domains, including chemical reactions, mass and energy movement, global weather prediction, and oil reservoir simulations. Molecular diffusion is the process by which solute molecules diffuse across a fluid. This happens when solute molecules randomly collide with fluid molecules, causing a flux from areas of higher concentration to areas of lower concentration. The advective term describes the bulk migration of solute particles in the direction of fluid flow at a rate equal to the fluid velocity. In addition to advective transport, molecular diffusion is another way that the solute spreads in porous media. Bear and Bachmat [3] state that the tortuosity of the medium and the diffusion coefficient of the particular solute in water determine the coefficient of molecular diffusion in an isotropic media. Interestingly, the rate of molecular diffusion advances even in the absence of fluid movement and is independent of groundwater velocity.

Many models and techniques have been developed over time to address the problem of groundwater contamination. To illustrate the transfer of contamination in biological, chemical, and radioactive processes, Younes [4] presented the Eulerian Lagrangian localized adjoint approach with a moving grid in 2005 for resolving nonlinear ARDE in one dimension. An analytical solution to an advection-diffusion equation with variable coefficients characterizing solute transport in porous media was given by Ahmed et al. [5]. Guerrero et al. [6] developed a method in 2009 to get analytic solutions for multi-species contamination transport controlled by sequential fractional-order equations in finite mediums by using traditional integral transform techniques. Enhancing the understanding and solution of complex dynamics related to advection, diffusion and interaction processes. The main goal of this work was by developing a new numerical approach to the nonlinear fractional ordering of ARDE spacetime:

$$\frac{d^{\alpha}u(x,t)}{dt^{\alpha}} = u(x,t)\frac{d^{\beta}u(x,t)}{dt^{\beta}} - v\frac{du(x,t)}{dx} + ku(x,t) \quad (1)$$
$$0 \le x \le 1, 0 < \alpha \le 1, 1 < \beta \le 2$$

with initial and boundary conditions as

$$u(x,0) = \psi_1(x), 0 \le x \le 1 \quad (2)$$

$$u(0,t) = \psi_2(t), t > 0 \quad (3)$$

$$u(1,t) = \psi_3(t), (t)t > 0 \quad (4)$$

The equation u(x, t), where the parameters A and ß stand for the corresponding fractional-order derivatives of time and space, represents the solute concentration in the fluid at position x and time t. The variable v represents the fluid's constant velocity in the x-direction, and the symbol k stands for the coefficient of the source/sink term, which is responsible for the solute's production or loss in the system. Moreover, the known functions Ψ1(x), Ψ2(t), and Ψ 3(t) indicate the initial distribution of solute concentration and the concentration at the medium's border points at any given time t. The fractional-order advection-reaction-diffusion equation (ARDE) is reduced to the standard ARDE with a nonlinear diffusion term when A = 1 and ß = 2. In the fluid domain, the solute concentration rises when this nonlinear diffusive factor is added in comparison to the linear diffusion equation. The diffusive term's positive exponent of u(x, t) causes slower diffusion rates than what is expected from a conventional linear diffusion scenario, which raises the fluid's solute concentration. In porous media systems, the contrast between slow and fast diffusion processes is highly relevant. Consequently, in comparison to a linear model, the existence of a nonlinear element in the diffusivity component is significant from a physical standpoint. This feature has motivated academics to work on nonlinear fractional-order porous media issues, highlighting how crucial it is to handle these kinds of difficulties in system dynamics.

The above-mentioned attributes and factors have spurred the creation of a new numerical technique intended to efficiently solve the time-fractional equation including the system's advection, diffusion, and reaction processes.

While solving Eq. (1), we have concentrated on using Dirichlet boundary conditions in our study.However, one can also employ well-posed and ill-posed Neumann and Cauchy boundary conditions for this purpose. In their research, Deng et al. [7] have thoroughly investigated the well-posedness of fractional diffusion models with various boundary conditions. Three types of spectrum methods—the collocation, Galerkin, and Tau methods—have reportedly been used to solve two-dimensional problems numerically in the literature.

By applying the chosen spectral approach to the solution, which is stated as a series of polynomials, such as ∑ aijØiØj, where Ø is a set of polynomials, the coefficients are obtained using spectral methods. The residues corresponding to partial differential equations (PDEs) in the collocation method must be zero at specified collocation points. Applying boundary conditions comes first in the Tau technique, which expands the residual function into a polynomial series. The Galerkin technique first selects basis functions that meet the initial and boundary criteria, and then it verifies that the residual is orthogonal to the selected basis functions.

In this work, we have attempted to solve the space-time fractional-order advection-reaction-diffusion problem, Eq. (1), by the use of its operational matrices and the Fibonacci collocation approach. Using example figures that correspond to particular circumstances, the effect of the reaction term on the solution profile under various parametric values of 'a' and 'ß'—which take into account the existence or absence of the advection term—is explained.

2. TF-ADR and SF-ADR MODEL SOLUTIONS

We obtained fractional ordinary differential equations by trans\forming the original models (1) and (3) in [14, 15, 16, 17] by means of Lie homologies. The original equations can be solved using the answers to these simplified equations. The FracSym package [18, 19, 20, 21], implemented in MAPLE, was used to find the Lie point symmetries. For fractional differential equations based on fractional Riemann-Liouville derivatives, this approach automatically finds symmetries. With the help of this suggested approach, answers that could be difficult to find by directly integrating the original model can be found.

Fractional partial differential equations (FPDEs) are often solved analytically or numerically in the literature. As an alternative to integrating an FPDE, We were able to solve a first-order ordinary differential equation at a low computational cost because it requires a simple initial condition. We summarize the main conclusions from [14, 15, 16, 17] in this section, to help identify solutions for the TF-ADR and SF-ADR simulations.

Based on the paper "A New Mathematical Process for Resolving the Fractional Time Equation of Advection, Diffusion and Reaction" the introduction to the chapter "Solutions to TF-ADR and SF-ADR Prototypes" has been rewritten in English.

2.1 The Model of Time Fraction

The infinitesimal Eq. (5) representing Lie symmetries accepted by the TF-ADR equation has been studied in [22].

$\xi 1 = 0\ \xi 2 = a1,\ \eta = \chi(t, x) + a2\ u,$ (5)

Meanng:

(\xi_1 = 0, \xi_2 = a_1) :

(\xi_1): This usually corresponds to the coefficient of the time variable ((t)) in a transformation. In this case, (\xi_1 = 0) indicates that there is no dependency on time in the symmetry or transformation being analyzed.

(\xi_2): This typically represents the coefficient of the spatial variable ((x)) in the transformation or symmetry. Here, (\xi_2 = a_1) means that the spatial variable (x) depends linearly on a parameter (a_1), which could be a scaling factor or a parameter related to the transformation.

\ )eta = \chi(t, x) + a_2 u:(

(\eta): This variable typically represents the transformed or modified dependent variable (for example, in a PDE or transformation context, it could be (u(t, x)), the primary variable of interest).

(\chi(t, x)): This is a function of the independent variables (t) (time) and (x) (space). It serves as a general term or a function that depends on the independent variables.

(u): This often refers to the solution or dependent variable in, say, a PDE. Scaling (u) by (a_2) means that (u) gets multiplied by a constant (or parameter) (a_2).

The expression (\eta = \chi(t, x) + a_2 u) combines the effect of (\chi(t, x)) and (u), showing that (\eta) incorporates both a general component ((\chi)) based on (t) and (x), and a contribution from (u), scaled by the parameter (a_2).

When the constraint is satisfied by the function

$\varphi = \varphi(t, x)$.

$$d_t^{\alpha} X - k_1 d_{xx} X + k_2 d_x X + \alpha_1 d_x f(.;\alpha) + d_u f(.;\alpha)(X + \alpha_2 u) - \alpha_2 f(.;\alpha) = 0 \quad (6)$$

If we impose A1 = 1 we get the following lie point transformation

$T = t,\ U = u(t,x)e^{-\alpha_2 x} - \int e^{-\alpha_2 x} X(t,x)dx$ we (7)
obtain the solution by the transformation Eq. (12),

(8)

$$u(t,x,\alpha)=e^{\alpha_2 x}\left(U(t)+\int e^{-\alpha_2 x}X(t,x)dx\right)$$

And the source term by integration of Eq. (11), $f(.;\alpha)=e^{\alpha_2 x}(\emptyset(t,U)-\int e^{-\alpha_2 x}(d_t^{\alpha}X(t,x)-k_1 d_{xx}X(t,x)+k_2 d_x X(t,x))d_x)$ (9)

The resolution and foundation terminology of the TF-ADR system (taking (lect= k_1 A^2 - k_2) and (Θ (t, U)) being a random purpose of its arguments) are represented by (u(t, x; α))) and (f(t, x, u; α)), respectively.

Therefore, equation (1) can be reduced to the succeeding slight nonlinear regular variance calculation by using transformation (6) and the preceding form of (f(. ; α)).

$$D_T^{\alpha}U(T)-a_2 \curlywedge U(T)+\emptyset\big(T,U(T)\big)=0 \quad (10)$$

By choosing an arbitrary function (Θ (T, U(T))) which modifies the basis terminology (f (t, x, u; α)) and the solution modules set via (6) the solution of equation (15) is defined.

In the particular scenario where (Ø (T, U(T)) = Ø *(T) + C_2U(T)), Ordinary fractional differential equation (15) can be expressed as:

$$D_T^{\alpha}U(T)+(c_2-a_2 \curlywedge)U(T)+\emptyset^*(T)=0 \quad (10)$$

And under non-vanishing initial conditions

$$[D_T^{\alpha-1}U(T)]_{T=0}=b_1 \quad (11)$$

Its precise solution, found in [8], is expressed in terms of the Mittag Leffler purpose,

$$U(T)=b_1 T^{a-1}E_{a,a}\big((a_2 \curlywedge -c_2)T^{a}\big)-\int_0^T (T-S)^{a-1}E_{a,a}\big((a_2 \curlywedge -c_2)(T-S)^{a}\big)\emptyset^*(S)dS \quad (12)$$

Were

$$E_{a,a}(t)=\sum_{k=0}^{\infty}\frac{t^k}{\mathcal{T}(a(k+1)}$$

And the basis terminology is given by

$$f(.;a)=c_2 u+k_1 d_x X(t,x)+e^{\alpha_2 x}\big(\emptyset^*(t)-\int e^{-\alpha_2 x}\big(c_2 X(t,x)+d_t^{a}X(t,x)-\curlywedge d_x X(t,x)\big)dx\big) \quad (13)$$

2.2 The Space Fractional Model

Specifically, the following SF–ADR equation (3) infinitesimals were found in [9]

ξ1 = ˜A1, ξ2 = 0, η = ˜χ(t, x) + ˜a2u

everywhere the parameters ˜a1 and ˜A2 and the function ˜χ = ˜χ(t, x) satisfy the restraint

$$d_t\tilde{X}-k_1 d_{xx}^{\beta+1}\tilde{X}+k_2 d_x\tilde{X}+\tilde{a}_1 d_t f(.;\beta)+\big(\tilde{X}+\tilde{a}_2 u\big)d_u f(.;\beta)-\tilde{a}_2 f(.;\beta)=0 \quad (14)$$

We obtain the following transformation through the assumed symmetries ˜A1 = 1

$$X=x,\qquad V=u(t,x)e^{-\widetilde{a_2}t}-\int e^{-\widetilde{a_2}t}\tilde{X}(t,x)dt \quad (15)$$

Thus, we obtained the precise solution to the fractal space problem

$$u(t,x;\beta)=e^{\tilde{a}2t}\big(V(x)+\int e^{-\tilde{a}2t}\tilde{X}(t,x)dt\big) \quad (16)$$

and, by

$$f(.;\beta)=e^{-\tilde{a}2t}\Big(\tilde{\emptyset}(x,V)-\int e^{-\tilde{a}2t}\big(d_t\tilde{X}(t,x)-k_1 d_x^{\beta+1}\tilde{X}(t,x)+k_2 d_x\tilde{X}(t,x)\big)dt\Big) \quad (17)$$

We can set (u(t, x; β)) and (f(t, x, u; β))) as the resolution and basis term of the SF-ADR system, using a stochastic function for its parameters as (φ (x, V)).

The SF-ADR equation (3) has also been simplified to the fractional ordinary differential equation through the transformation in equation (15) and the preceding formulation of (f(. ; β)):

$$-k_1 D_X^{\beta+1}V(X)+k_2 D_X V(X)+\tilde{\emptyset}\big(X,X(X)\big)+\tilde{a}_2 V(X)=0 \quad (18)$$

The solution of equation (24), which can be customized with the right choice of ˜χ = ˜χ (t, x), defines the random-roles φ (X, V(X)). This, in line, determines the categories of solutions and the basis terminology (f(t, x, u; β))

In particular, setting

$$\tilde{\emptyset}\big(X,V(X)\big)=-k_1 D_X\emptyset^{**}(X)-\tilde{a}_2 V(X) \quad (19)$$

we get

$$\boldsymbol{D_X^{\beta}V(X)} - \boldsymbol{\frac{k_2}{k_1}V(X)} + \emptyset^{**}\boldsymbol{(X)} = 0 \quad (20)$$

Whose precise answer, with respect to non-vanishing boundary conditions

$$\left[D_X^{\beta-1}V(X)\right]_{X=0} = \tilde{b}_1$$

is the following

$$V(X) = \tilde{b}_1 X^{\beta-1} E_{\beta,\beta}\left(\frac{k_2}{k_1}X^{\beta}\right) - \int_0^X (X - S)^{\beta-1} E_{\beta,\beta}\left(\frac{k_2}{k_1}(X-S)^{\beta}\right)\emptyset^{**}(S)dS \quad (21)$$

Where

$$E_{\beta,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\mathcal{T}(\beta(k+1))}$$

is the Mittag Leffler function [8] and the source terminology states.

$$f(\cdot;\beta) = -\tilde{a}_2 u$$
$$-e^{\tilde{a}2t}\left(k_1\emptyset_x^{**}(x) + \int e^{-\tilde{a}2t}\left(-\tilde{a}_2\tilde{X} + d_t\tilde{X}(t,x) - k_1 d_x^{\beta+1}\tilde{X}(t,x) + k_2 d_x\tilde{X}(t,x)\right)dt\right) \quad (22)$$

We denote the resolution and the basis term for the SF–ADR system as u(t, x; β) and f(t, x, u; β), respectively, with φ (x, V) as a random-purpose of its arguments. The SF–ADR problem (3) was reduced to the subsequent slight regular variance calculation by using conversion (21) and the original form of (f.; β) Rewrite to suit the heading: A new computational technique for solving the time-fractional equation of advection, distribution, and response

$$-k_1 D_X^{\beta+1}V(X) + k_2 D_X V(X) + \tilde{\emptyset}\big(X, V(X)\big) + \tilde{a}_2 V(X) = 0 \quad (23)$$

Solving calculation (23), the solution is defined by choosing arbitrary functions φ˜ (X, V(X)) where with the appropriate selection of ˜ χ = ˜ χ (t, x) assigns the basis term f(t, x, u; β) and solution modules.

Specifically, rewrite to suit the heading: A novel numerical technique for solving the time fractional equation of diffusion, and reaction

$$\tilde{\emptyset}\big(X, V(X)\big) = -k_1 D_X \emptyset^{**}(X) - \tilde{a}_2 V(X)$$

we get (24)

$$D_X^{\beta}V(X) - \frac{k_2}{k_1}V(X) + \emptyset^{**}(X) = 0$$

Whose precise answer, with respect to non-vanishing boundary conditions

$$\left[D_X^{\beta-1}V(X)\right]_{X=0} = \tilde{b_1} \quad (25)$$

is the following

$$V(X) = \tilde{b}_1 X^{\beta-1} E_{\beta,\beta}\left(\frac{k_2}{k_1}X^{\beta}\right) - \int_0^X (X - S)^{\beta-1} E_{\beta,\beta}\left(\frac{k_2}{k_1}(X-S)^{\beta}\right)\emptyset^{**}(S)dS \quad (26)$$

$$f(.;\beta) = -\tilde{a}_2 u$$
$$-e^{\tilde{a}_2 t}\left(k_1\emptyset_x^{**}(x) + \int e^{-\tilde{a}_2 t}(-\tilde{a}_2\tilde{X} + d_t\tilde{X}(t,x) - k_1 d_x^{\beta+1}\tilde{X}(t,x) + k_2 d_x\tilde{X}(t,x)\right)dt) \quad (26)$$

3. TIME and SPACE FRACTIONAL MODEL SOLUTIONS

This division examines the TSF–ADR equation's Lie symmetries

$$d_t^a u(t,x) - k_1 d_{xx}^{\beta+1}u(t,x) + k_2 d_x u(t,x) + f(t,x,u;a,\beta) = 0,\ \ 0 < a,\beta \le 1 \quad (27)$$

The infinitesimal generators listed below describe the Lie symmetries that are admitted by (27)

ξ1 = 0, ξ2 = 0, η = ¯ χ(t, x) + ¯ a2u (28)

Everywhere the constraint is satisfied by the purpose χ = ¯ χ(t, x)

$$d_t^a\bar{X} - k_1 d_{xx}^{\beta+1}\bar{X} + k_2 d_x\bar{X} + (\bar{X} + \bar{a}_2 u)d_u f(.;a,\beta) - \bar{a}_2 f(.;a,\beta) = 0 \quad (29)$$

The method described in [9] does not work since the symmetries recognized by equation (27) have infinitesimal (28) preventing the transformation that transforms the TSFADR equation into a fractional normal equation. Based on the data presented in the above section we present a substitute method to get the mathematical resolution. And the analytical prototypical of TSF-ADR.

Let us now examine u(t, x; α; β) as the resolution to the TSF–ADR equation provided by

$$u(t,x;a;\beta) = A\,u(t,x;a) + B\,u(t,x;\beta)$$

the solutions produced by the transformations (12) and (20) are combined linearly

$$u(t,x;a) = e^{a_2 x}U(t) + e^{a_2 x}\int e^{-a_2 x}X(t,x)dx$$

$$u(t,x;\beta) = \quad e^{\tilde{a}_2 t}V(x) + e^{\tilde{a}_2 t}\int e^{-\tilde{a}_2 t}X(t,x)dx \quad (30)$$

where, we put χ = ˜ χ, consequently we take

$$u(t,x;a;\beta) = A\, e^{a_2 x}U(t) + A\, e^{a_2 x}\int e^{-a_2 x}X(t,x)dx + B\, e^{\tilde{a}_2 t}V(x) + B\, e^{\tilde{a}_2 t}\int e^{-\tilde{a}_2 t}X(t,x)dt \quad (31)$$

where the solutions to the simplified equations (14) and (23), respectively, are U(t) and V (x)

We get that the resolution of the TSF-ADR prototypical when the basis terminology f(t, x, u; α; β) is a lined amalgamation of 2 roles (29)

$$\mathrm{f}(\mathrm{t},\mathrm{x},\mathrm{u};\ \alpha;\ \beta) = Af_1(\mathrm{t},\mathrm{x},\mathrm{u};\ \alpha;\ \beta) + Bf_2(\mathrm{t},\mathrm{x},\mathrm{u};\ \alpha;\ \beta)$$

given by

$$f_1(\mathrm{t},\mathrm{x},\mathrm{u};\ \alpha;\ \beta) = e^{a_2 x}\Big(\emptyset\big(t,U(t)\big) - \int e^{-a_2 x}\big(d_t^a X(t,x) + k_2 d_x X(t,x)\big)d_x\Big) + k_1 d_x^{\beta+1}\Big(e^{a_2 x}\int e^{-a_2 x}X(t,x)dx\Big) + k_1 a_2\Big(a_2 e^{a_2 x} - x^{-\beta}E_{1,1} - \beta(a_2 x)\Big)U(t) \quad (32)$$

$$f_2(\mathrm{t},\mathrm{x},\mathrm{u};\ \alpha;\ \beta) = e^{\tilde{a}_2 t}\Big(\tilde{\emptyset}\big(x,V(x)\big) - \int e^{-\tilde{a}_2 t}\Big(-k_1 d_x^{\beta+1}X(t,x) + k_2 X_x(t,x)\Big)d_t\Big) - d_t^a\Big(e^{\tilde{a}_2 t}\int e^{-\tilde{a}_2 t}X(t,x)dt\Big) + \Big(\tilde{a}_2 e^{\tilde{a}_2 t} - t^{-a}E_{1,1} - a(\tilde{a}_2 t)\Big)V(x)$$

We observe that terms including β are smaller than f1(t, x, u; α, β) equals f (t, x, u; α) (14) and terms involving β are less than f2(t, x, u; α, β) equals f (t, x, u; β) (23). In fact, if c = 1, we have

$$f1\ (t,x,u;\ \alpha;\ \beta)|_{\beta=1} \quad (33)$$
$$= e^{a_2 x}\Big(\emptyset\big(t,U(t)\big) - \int e^{-a_2 x}\big(d_t^a X(t,x) + k_2 d_x X(t,x)\big)d_x\Big) - k_1\Big(e^{a_2 x}\int e^{-a_2 x}d_{xx}X(t,x)dx\Big)$$

and if α = 1

$$f2\ (t,x,u;\ \alpha;\ \beta)|_{\alpha=1} = e^{\tilde{a}_2 t}\Big(\tilde{\emptyset}\big(x,V(x)\big) - \int e^{-\tilde{a}_2 t}\Big(-k_1 d_x^{\beta+1}X(t,x) + k_2 X_x(t,x)\Big)d_t\Big) - \Big(e^{\tilde{a}_2 t}\int e^{-\tilde{a}_2 t}d_t X(t,x)dt\Big)$$

Specifically, using the definitions in (2) and (4), for β = 1 and α = 1, $\partial^t_\alpha = \partial t$ and $\partial x^{\beta+1} = \partial_{xx}$, respectively. Next, we have

$$f_1(t,x,u,a,\beta)|\beta = 1 = f(t,x,u,a)$$
$$f_2(t,x,u,a,\beta)|a = 1 = f(t,x,u,\beta)$$

That is, we can use these resolutions to catch a solution for the TSF model. ADR, if f(t, x, u, α) and f(t, x, u, β) are the source terminologies for the resolutions of SF-ADR and TF-ADR which are known to us.

4. FROM the STANDARD ADR MODEL to the TSF-ADR MODEL

Appropriate assumptions were made for Any functions χ(t, x) , φ (x, V), φ (t, U) and based on the results from previous sections. As a result, a specific physical problem might be defined with an appropriate source term, enabling the extraction of the matching traditional ADR model as the TSFADR model's limit. Next, in order to obtain, we choose χ(t, x), and φ(t, U), φ˜ (x, V) in section

$$f(t,x,u;a)|_{a=1} = f(t,x,u;\beta)|_{\beta=1} = f(t,x,u) \quad (34)$$

so that the terms of the combination in linear form (31) fulfill the following as a result of (32)

$$f1(t,x,u;a,\beta)|_{a=1,\beta=1} = f(t,x,u)$$
$$f2(t,x,u;a,\beta)|_{a=1,\beta=1} = f(t,x,u)$$

Finally, we obtain assuming A + B = 1 (35)

$$f(t,x,u;a,\beta)|_{a=1,\beta=1} = f(t,x,u)$$

This means that the basis term of TSF-ADR is equivalent to the role f(t, x, u) of the conventional ADR system when α=1 and β= 1.

$$dt_u(t,x) - k_1 d_{xxu}(t,x) + k_2 d_{xu}(t,x) + f(t,x,u) = 0 \quad (36)$$

Considering that we do not recognize the exact formula of the resolution (29), (it is obtained through the set of linear solutions (30) for α = β = 1) and therefore we are unable to show the relationship amongst the resolution of the TSF-ADR prototypical and the resolution of the ADR system. In the subsequent analysis, we employ a numerical method to demonstrate that the TSF–ADR model's solution (29) has the same limit as the ADR model's solution.

5. THE NUMERICAL METHOD

Here, we provide the numerical outcomes of applying the suggested method to get TSF-ADR model answers that are connected to precise ADR model solutions. The method was presented in [10, 11], where a number of numerical tests were used to confirm the procedure's accuracy and efficiency with respect to the answers produced for the SF-ADR and TF-ADR models. Using the same methods in the current work, numerical solutions for the SF – ADR , TF – ADR models were derivedwhich allows mathematical resolutions of the TSF-ADR system to meet the required conditions (32).

We now provide a brief overview of the procedure. We begin by independently solving the two simplified equations (14) and (23). We apply transformations (12) and (20) to the numerical solutions we find satisfactory to develop estimated resolutions for the SF-ADR system (3) and the TF-ADR system (1). The two discovered numerical solutions are then combined to get the mathematical resolution for the TSF-ADR system (5), as mentioned (29). Finally, we show that when α → 1 and β → 1 the solutions of the TSF-ADR model typically agree with the ADR model.

We decided to examine Caputo's method, which contains the numeral-order results of the indefinite purposes in minimum time at their limit values, i.e. has the fundamental benefit of

having initial conditions similar to those of integer-order differential equations.

We go over how Caputo's fractional derivative is explained.

$$* D_T^a U(T) = \frac{1}{\mathcal{T}(1-a)} \int_0^T \frac{1}{(T-S)^a} \frac{d}{dS} U(S) dS \quad (37)$$

and its connection to the fractional derivative of Riemann-Liouville

$$* D_T^a U(T) = D_T^a \big(U(T) - U(0)\big), \qquad 0 < a < 1$$

U(0) is the starting point. The fractional time equation (15) can be expressed, based on the relationship amid the Riemann-Liouville and Caputo results, in the following way:

$$D_T^a U(T) + \frac{U(0)}{\mathcal{T}(1-a)T^a} - a_2 \setminus U(T) + \emptyset\big(T, U(T)\big) = 0 \quad 0 < a < 1 \quad (38)$$

The RL derivative and the Kabuto derivative are equal under homogeneous starting conditions, where the typical factor of the Kabuto derivatives is ∗DαT [8].

We also reformulated the fractional space equation (23) in terms of the Caputo resultant using the same assumptions

$$k_1^* D_X^{\beta+1} V(X) + \frac{V(0)}{\mathcal{T}(1-\beta)X^\beta} - k_2 D_X V(X) - \tilde{a}_2 V(X) - \tilde{\emptyset}\big(X, V(X)\big) = 0 \quad (39)$$

When it is not possible to determine the logical resolutions U(T) and V (X), a numerical method must be employed. In this instance, we offer a numerical scheme, the 2nd direction contained trapezoidal process (TR), It is frequently used to solve linear and nonlinear fractional ordinary differential equations as it is an oversimplification of the standard implied trapezoid technique on fractional ordinary differential equations (FODEs). See the papers [12, 13] for the technical specifics of the suggested approach. A nonlinear equation must be solved each time fractional ordinary differential equations are integrated using the numerical approach.Thus, a nonlinear equation solution methodology, such as the conventional Newton method, must be used.

One of the most widely used iterative methods is Newton's. It requires evaluating the Jacobian matrix at each iteration, which can be done analytically if the processing cost is too high. Alternatively, it can be computed numerically. It is analytically assessed for each example in this work, and numerical answers are achieved with an equal number of iterations. Using MatLab software, the suggested numerical technique is implemented on an Intel Core i5.

In order to test the suggested method, we appropriately select the arbitrary functions in the following. As a result, we can obtain analytical solutions and compare them with numerical solutions.

We set

$$\varphi(t,U) = \varphi * (t) + c_2 U(t)\tilde{\varphi}(x,V) = -k_1 D_x \varphi^{**}(x) - \tilde{a}_2 V(x) \quad (40)$$

for the analytical solution to equations (36) and (37) to be provided in Mittag-Leffler functions in terms (17) and (26). As a result, we also get the TSF-ADR equation's analytical solution (29) (5).
The presented numerical test results confirm the efficacy and dependability of the suggested technique based on the assessment of estimated and precise resolutions for a variety of grid facts and the fractional order standards of the results α and β, which showed a high degree of accuracy.

6. FRACTIONAL-ORDER ADVECTION REACTION DIFFUSION EQUATION SOLUTION FOR SPACE–TIME

The authors have attempted to use the suggested numerical technique in Sect. 5 after confirming its efficacy, efficiency, and correctness. The method will be used to solve the space-time fractional-order ARDE under the initial condition

u(x,0) = x(1-x)

and boundary conditions

u(0,t) = 0, u(1,t)=0

Now, to solve the ARDE using the proposed numerical method let us approximate u(x, t) by Fibonacci polynomial as

$$u(x,t) \cong \emptyset^T(x) C \emptyset(t) \quad (41)$$

Where

$$C = \begin{pmatrix} c_{11} & c_{12} \; \cdots & \cdots & c_1 n + {}_1 \\ c_{21} & c_{22} \; \cdots & \cdots & c_1 n + {}_1 \\ \vdots & \vdots & \ddots & \vdots \\ cn_1 & cn_2 & \cdots & cn + {}_1 n + {}_1 \end{pmatrix}_{(n+1)\times(n+1)}$$

$$\begin{aligned} R(x,t) &= t^{(-a)} \emptyset^T(x) C M^a \emptyset(t) \\ &- x^{-\beta} (\emptyset^T(x) C \emptyset(t)) (\emptyset^T(x) \left(M^\beta\right)^T C \emptyset(t) \\ &+ v\left(\emptyset^T(x) (M^1)^T C \emptyset(t)\right) - k(\emptyset^T(x) C \emptyset(t)) \end{aligned}$$

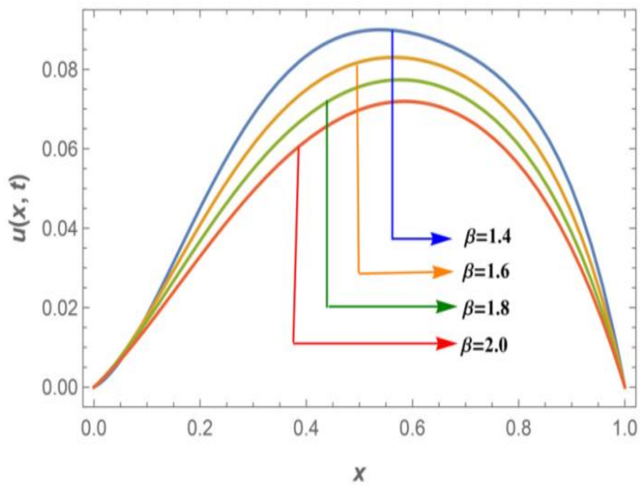

Figure 1. Variations of the conservative system's u(x, t) versus x at t = 0.6 when v = 0.2

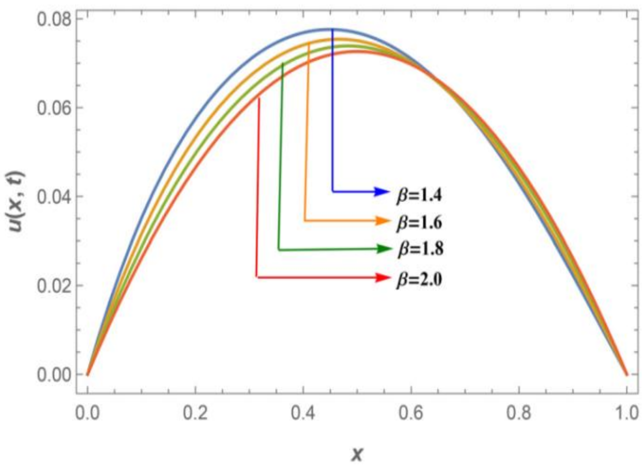

Figure 2. Variations of the conservative system's u(x, t) vs x at t = 0.6 for v = 0

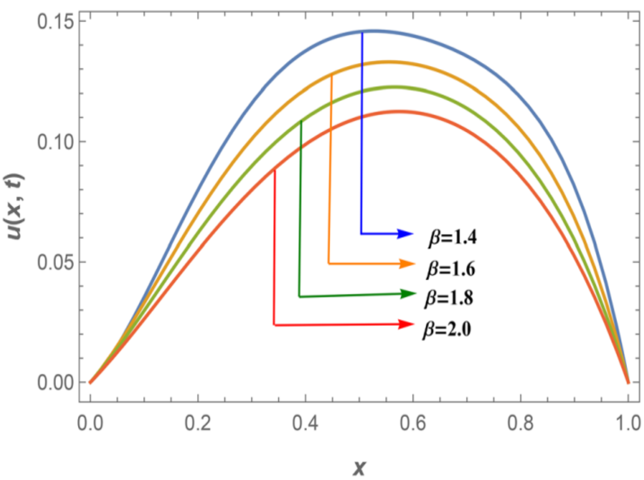

Figure 3. Variations of the nonconservative system's u(x, t) vs x at t = 0.6 for v = 0

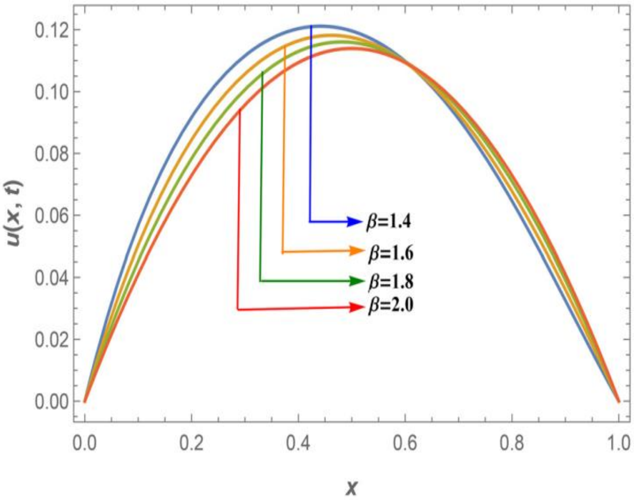

Figure 4. Variations of the nonconservative system's u(x, t) vs x at t = 0.6 for v = 0.2

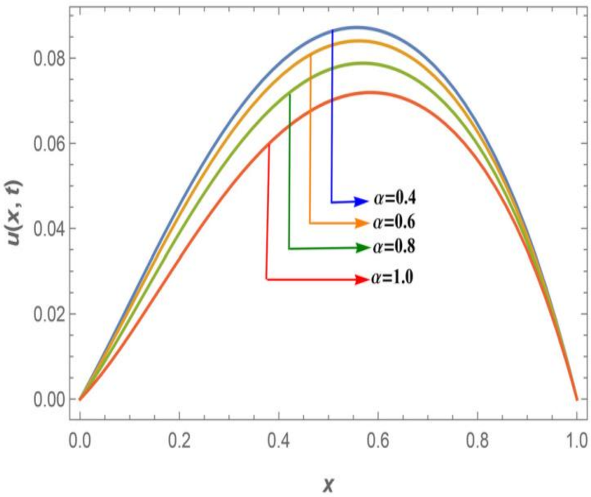

Figure 5. Variations of the conservative system's u(x, t) vs x at t = 0.6 for v = 0.2

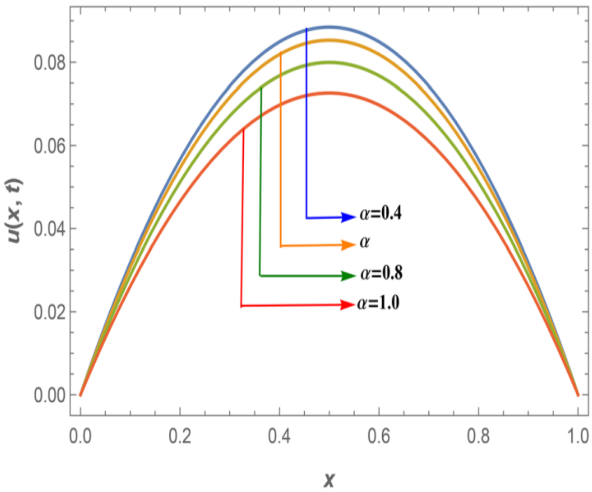

Figure 6. Variations of the conservative system's u(x, t) vs x at t = 0.6 for v = 0

h = 1, 2, 3,..n. To collocate the boundary conditions, x = 1 and t = 1 must be included in addition to these points. At last, we have an algebraic equation system of $(n+1)^2$ numbers of unknowns, which can be solved using Newton's method.

MATHEMATICA software is utilized to acquire numerical results for n = 7.

7. RESULTS and DISCUSSION

This section's Figs. 1–6 show the numerical solute concentration values in the presence and absence of advection and reaction terms, along with changes in the temporal parameter a while maintaining the spatial parameter b fixed, as well as changes in b for a fixed a.

In order to demonstrate the impact of the advection term, the changes of u(x, t) versus x at t = 0.6 are shown in Figs. 1-2 and 3–4 for the conservative system (k = 0) and the nonconservative system (k = 1), respectively, maintaining a = 1 and ß = 1.4(0.2) 2.0. It can be observed that in both systems when there is an advection term present, u(x, t) drops as b increases, but subsequently the converse happens.

Moreover, it is observed that in the case of a nonconservative system, the sink term (k = 1) causes damping. Once more, for both conservative and nonconservative systems, the solute concentration translations are easily observed with the fluid velocity (v = 0.2) and without any alteration to the curves' slopes.

The changes of u(x, t) for conservative and nonconservative systems, assuming ß = 1 and a = 0.4(0.2) 1.0, are depicted in Figures 5–6 with x at t = 0.6. The figures have resemblance to earlier instances, with the exception of variations in overshoots.

8. CONCLUSION

In this study, based on the outcomes of the SF-ADR and TF-ADR models, we describe a strategy for solving the TSF-ADR model. By bearing in mind the lined fusion of the resolutions of the SF-ADR and FT-ADR systems, we can then determine the model solutions.

We use a numerical method that is frequently used to solve lined and nonlinear models of small regular variance calculations. Mathematical testing and error exploration provide closeness classification as evidence of the correctness, efficacy and dependability of the recommended approach.

9. REFERENCES

1. Koc¸, A. B., C¸akmak, M., Kurnaz, A., and Uslu, K., 2013, "A New Fibonacci Type Collocation Procedure for Boundary Value Problems," Adv. Differ. Equa- tions, 2013(1), p. 262.
2. Abd-Elhameed, W. M., and Youssri, Y. H., 2016, "A Novel Operational Matrix of Caputo Fractional Derivatives of Fibonacci Polynomials: Spectral Solutions of Fractional Differential Equations," Entropy, 18(10), p. 345.
3. Bear, J., and Bachmat, Y., 1967, "A Generalized Theory on Hydrodynamic Dis-persion in Porous Media," IASH Symposium on Artificial Recharge and Man- agement of Aquifers, Vol. 72, IASH Publ. Int. Union Geod. Geophys., Haifa, Israel, pp. 7–16.
4. Younes, A., 2005, "A Moving Grid Eulerian Lagrangian Localized Adjoint Method for Solving One-Dimensional Nonlinear Advection-Diffusion-Reaction Equations," Transp. Porous Media, 60(2), pp. 241–250.
5. Ahmed, M., Zainab, Q. U. A., and Qamar, S., 2017, "Analysis of One-Dimensional Advection–Diffusion Model With Variable Coefficients Describing Solute Trans- port in a Porous Medium," Transp. Porous Media, 118(3), pp. 327–344.
6. Guerrero, J. S. P., Skaggs, T. H., and Van Genuchten, M. T., 2009, "Analytical Solution for Multi-Species Contaminant Transport Subject to Sequential First-Order Decay Reactions in Finite Media," Transp. Porous Media, 80(2), pp. 373–387.
7. Deng, W., Li, B., Tian, W., and Zhang, P., 2018, "Boundary Problems for the Fractional and Tempered Fractional Operators," Multiscale Model. Simul., 16(1), pp. 125–149.
8. I. Podlubny. Fractional Differential Equations: an introduction to fracti- onal derivatives, fractional differential equations, some methods of their solution and some of their applications, Academic Press, San Diego, (1999).
9. A. Jannelli, M. Ruggieri and M. P. Speciale, Exact and Numerical Solutions of Time fractional Advection–diffusion Equation with a Nonlinear Source Term by means of the Lie Symmetries, Nonlinear Dynamics, (2018); doi: 10.1007/s11071-018-4074-8.
10. A. Jannelli, M. Ruggieri and M. P. Speciale, Analytical and Numerical Solutions of Fractional Type Advection-diffusion Equation, AIP Conference Proceedings 1863, 530005 (2017). doi: 10.1063/1.4992675.
11. A. Jannelli, M. Ruggieri and M. P. Speciale, Analytical and Numerical Solutions of Fractional Type Advection-diffusion Equation, AIP Conference Proceedings 1863, 530005 (2017). doi: 10.1063/1.4992675.
12. K. Diethelm. The Analysis of Fractional Differential Equations. Springer-Verlag Berlin Heidelberg, (2010).
13. R. Garrappa. Trapezoidal methods for fractional differential equations: Theoretical and computational aspects, Math. and Comput. in Simulation, 110, 96–112, (2015).
14. Heydari, Mohammad Hossein Et Al. "Numerical Treatment of The Space–Time Fractal–Fractional Model of Nonlinear Advection–Diffusion–Reaction Equation Through The Bernstein Polynomials." Fractals (2020): 2040001.
15. Jannelli, Alessandra et al. "Numerical solutions of space-fractional advection-diffusion equations with nonlinear source term." Applied Numerical Mathematics 155 (2019): 93-102.
16. Dwivedi, Kushal Dhar et al. "Numerical Solution of Nonlinear Space–Time Fractional-Order Advection–Reaction–Diffusion Equation." Journal of Computational and Nonlinear Dynamics 15 (2020): n. pag.
17. Scotton, Jaque Willian Et Al. "Simulação Numérica Da Advecção-Difusão Unidimensional Empregando Um Modelo Com Derivadas Fracionárias." (2020).
18. Jefferson, G. F. and John Carminati. "FracSym: Automated symbolic computation of Lie symmetries of fractional differential equations." Comput. Phys. Commun. 185 (2014): 430-441.
19. Sethukumarasamy, K. et al. "On Lie Symmetry Analysis of Certain Coupled Fractional Ordinary Differential Equations." Journal of Nonlinear Mathematical Physics (2021): n. pag.
20. Al-deiakeh, Rawya et al. "Lie symmetry analysis, explicit solutions, and conservation laws of the time-fractional Fisher equation in two-dimensional space." Journal of Ocean Engineering and Science (2021): n. pag.
21. Singla, Komal and Rajesh Kumar Gupta. "Generalized Lie symmetry approach for fractional order systems of differential equations. III." Journal of Mathematical Physics 58 (2017): 061501.
22. Fujita, Yasuhiro. "Integrodifferential Equation Which Interpolates The Heat Equation And The Wave Equation I(Martingales And Related Topics)." (1989).

Arabic Abstract

تم استخدام تحويلات نظرية لاي في هذه الدراسة لتوفير أجوبة رقمية وتحليلية على السواء لمعادلات التفاعل الجزئي - الانتشار - الالتصاق لكل من الزمان والمكان. و إذا كانت الأرصدة المسموح بها من خلال حسابات الهدف تسمح بتحديد تحويلات نظرية لاي، فهنا يمكننا خفض حسابات الفرق الكسرية الطفيفة إلى حسابات الفرق العادية التي تحتوي على الكسور. ونقترح نهجاً مختلفاً لإيجاد الإجابات العددية والتحليلية بدءاً من الإجابات العددية والتحليلية الموجودة في نموذج لفراكتل الزماني المكاني في تفاعل الالتصاق والانتشار. وبالنسبة للنتائج الأخيرة بشأن معادلة التفاعل بين الالتصاق والانتشار التي حصل عليها المؤلفون. فان معادلات تفاعلات التصاق ونشر منفصلة لمتغيرات وقتية ومكانية جزئية. والدقة الممتازة للنهج المقترح تجعله أداة مفيدة لحل فئة واسعة من مشاكل المعادلة التفاضلية الكسرية.
وتظهر النتائج العددية فعاليتها وامكانياتها وقابليتها للتطبيق